Exercise Sheet 5

1.
   - Note that $\frac{1}{(z-a)(z-b)} = \frac{1}{b-a}(\frac{1}{z-a} - \frac{1}{z-b})$. Hence:

$$\int_{\gamma_R} \frac{f(z)dz}{(z-a)(z-b)} = \frac{1}{a-b}\left(\int_{\gamma_R} \frac{f(z)dz}{(z-a)} - \int_{\gamma_R} \frac{f(z)dz}{(z-b)}\right) = $$
$$= \frac{2\pi i}{a-b}(f(a) - f(b)).$$

   The last equality follows from Cauchy's theorem.

   - Since f is bounded, hence $|f(z)| < M$, for all $z \in \mathbb{C}$. Now note that:

$$\left|\int_{\gamma_R} \frac{f(z)dz}{(z-a)(z-b)}\right| \leq 2\pi R \max_{z\in\gamma_R} \left|\frac{f(z)dz}{(z-a)(z-b)}\right| \leq \frac{2\pi RM}{|R-|a|||R-|b||}.$$

   This clearly goes to 0 as R goes to infinity.

   - By the first item the integral is constant, whereas by the second, it goes to 0 as R goes to ∞. This can only happen if the constant is 0. Hence $f(a) = f(b)$, for any arbitrary pair of distinct points. Thus f is constant.

2. Write $f = u + iv$ and assume $|u| \leq M$. Since f is entire so is $g(z) = e^{f(z)}$. Now note that $|g(x+iy)| = |e^{u(x,y)+iv(x,y)}| = e^{u(x,y)} \leq e^M$. By Liouville's theorem $g(z) = g$ is constant. This implies that the image of f is $\log|g| + i\arg(g) + 2\pi ik$. Since f is continuous on $\mathbb{C}$, the image is precisely one point, thus f is constant.

   Another way to see it is derivating e^f. Since e^f is constant, $f'e^f = 0$, but $e^f \neq 0$, hence $f' = 0$.

3. Since f is entire and $f(\mathbb{C}) \subset \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, the function $h(z) = g(f(z))$, where g is the branch of the square root specified in the hint, is entire. However note that the image of g is only a half-plane (since $-\pi < \theta < \pi$, the image of g is in the right half-plane). Conclude that h is constant, for otherwise, its image is dense in $\mathbb{C}$. But $h^2 = f(z)$, hence it is constant.

   For those of you that have not seen dense sets and the above result, the only thing we need is to show that if F is an entire function and there exists an $r > 0$, such that for every $z \in \mathbb{C}$, $F(z) \notin D_r(0)$, then F is constant. To prove it note that this hypothesis implies that for every $z \in \mathbb{C}$, $|F(z)| \geq r$. In particular F is never zero. Hence $\frac{1}{F}$ is entire, but $|\frac{1}{F(z)}| \leq \frac{1}{r}$, for every $z \in \mathbb{C}$. Hence F is constant.

   Now just take $F(z) = h(z) + 1$, $\mathrm{Re}(F(z)) > 1$, for every $z \in \mathbb{C}$ and hence the image of F misses the unit disc around 0. By what we've just shown, F is constant, but hence h is constant.

4.
   - Let f be a function holomorphic in the unit disc, such that $f(1/n) = \frac{(-1)^n}{n^2}$. Then by continuity of f, we get that $f(0) = 0$. Note that if $g(z) = -z^2$ and $h(z) = z^2$, then for n even $f(1/n) = h(1/n)$, whereas for n odd $f(1/n) = g(1/n)$. Since the accumulation point of both $\frac{1}{2n}$ and $\frac{1}{2n+1}$ is 0, by the uniqueness theorem $f(z) = g(z) = h(z)$, but $g(z) \neq h(z)$.
   - If $f(\frac{1}{n}) = \frac{1}{n+1}$, let $g(z) = \frac{z}{z+1}$. By continuity $f(0) = 0$ and so is $g(0) = 0$. Note that $f(\frac{1}{n}) = g(\frac{1}{n})$ and hence by the uniqueness theorem they coincide on the disc. Hence $f(\frac{i}{2}) = g(\frac{i}{2}) = \frac{i}{i+2} = \frac{i(i-2)}{5} = -\frac{1+2i}{5}$.
   - Not necessarily. Take $f(z) = \sin(\pi z)$, then its zeroes are precisely the integers, however it is not the zero function. It does not contradict the uniqueness theorem since the zeroes don't have an accumulation point in $\mathbb{C}$.

5. Since f is entire, we can write $f(z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{n!} z^m$, this series converges everywhere in $\mathbb{C}$. By Cauchy's formula:

$$\frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z^{m+1}}.$$

Where we take γ to be a circle around 0 of radius $\rho > R$. Hence, for $m > n$:

$$\left| \frac{f^{(m)}(0)}{m!} \right| \leq \frac{1}{2\pi} 2\pi\rho \max_{z \in \gamma} \left| \frac{f(z)dz}{z^{m+1}} \right|.$$

Now since $\rho > R$, we know that $|f(z)| \leq |z|^n = \rho^n$. Therefore:

$$\left| \frac{f^{(m)}(0)}{m!} \right| \leq \rho^{n-m}.$$

The right hand side goes to 0 as ρ approaches infinity, whereas the left hand side is constant, hence we conclude that for every $m > n$, $f^{(n)}(0) = 0$. This, however, implies that the series is in fact a polynomial.

6. Set $f(z) = z^a$. Note that if a is an integer than on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, f coincides with z^n. Since $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is open, using the uniqueness theorem we can identify the two functions. Hence this function is entire for a a non-negative integer, analytic in $\mathbb{C} \setminus \{0\}$ for a a negative integer and analytic (as the composition of two analytic functions) in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

   Now in general $f'(z) = \frac{a}{z} e^{a \log(z)}$, now by the above statement $z^{-1} = e^{-\log(z)}$, hence $f'(z) = ae^{(a-1)\log(z)} = az^{a-1}$.

   Now set $g(z) = (1+z)^a$, hence $g^{(n)}(0) = a(a-1)\cdots(a-n+1)$. Denote $b_n(a) = \frac{a(a-1)\cdots(a-n-1)}{n!}$. Therefore the Taylor expansion is:

$$g(z) = \sum_{n=0}^{\infty} b_n(a) z^n.$$

   Note that if a is a non-negative integer, then there exists an N, such that for all $n > N$, $b_n(a) = 0$. Hence this series is in fact a polynomial and thus the radius of convergence is infinite. If a is not a non-negative integer, then we use the D'Alambert test:

$$\left| \frac{b_{n+1}(a)}{b_n(a)} \right| = \left| \frac{a-n}{n+1} \right| \to 1.$$

   Hence the radius of convergence is 1. Which fits since in this case the function has a singularity at -1.

7. - Denote U the interior of γ. We first prove that F_1 is continuous. Let a be a point in the interior of γ, then there is an $r > 0$ (open set), such that $D_r(a) \subset U$ and for every $z \in D_r(a)$ and every $t \in [a, b]$, we have $|\gamma(t) - z| > r$. Let $z \in D_r(a)$, we compute:

$$|F_1(z) - F_1(a)| = \left| \int_{\gamma} \frac{\varphi(w)dw}{(w-z)} - \int_{\gamma} \frac{\varphi(w)dw}{(w-a)} \right| = \left| \int_{\gamma} \frac{\varphi(w)(z-a)dw}{(w-z)(w-a)} \right| \leq$$
$$\leq L(\gamma) M \frac{1}{r^2} |z-a|.$$

     Now for every $\epsilon > 0$ you can always find r small enough, satisfying the above requirements and such that the above expression is less than ϵ.

Now the above computation also shows that:

$$\frac{F_1(z) - F_1(a)}{z-a} = \int_\gamma \frac{\varphi(w)dw}{(w-z)(w-a)}.$$

This tends to $F_2(a)$, when $z \to a$. Now proceed by induction.

- We do this iteratively. Set $f_0(z) = f(z)$ and $f_n(z) = \frac{f_{n-1}(z) - f_{n-1}(a)}{z-a}$. Then note that $\lim_{z\to a}(z-a)f_1(z) = 0$. Furthermore f_1 is continuous and $\lim_{z\to a} f_1(z) = f'(a)$. Hence f_1 is analytic, in Ω. Similarly one proceeds for every $f_n(z)$. Therefore $f(z) = f(a) + (z-a)f_1(z)$, $f_1(z) = f_1(a) + (z-a)f_2(z)$, etc.

  From this we get that:

  $$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \ldots + (z-a)^N f_N(a) + (z-a)^{N+1} f_{N+1}(z).$$

  Differentiating this expression k times and setting $z = a$, we get that $f_k(a) = \frac{1}{k!}f^{(k)}(a)$. This is true for $1 \leq k \leq N$.Now $f_{N+1}(z)$ is analytic, hence we can apply the Cauchy formula to it with our contour γ and get:

  $$f_{N+1}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_{N+1}(w)dw}{w-z}.$$

  Plugging in the equation derived above, we get:

  $$f_{N+1}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_(w)dw}{(w-z)(w-a)^{N+1}} - \frac{1}{2\pi i}\sum_{k=0}^{N}\int_\gamma \frac{f^{(k)}(a)dw}{(w-z)(w-a)^{N+1-k}}.$$

  Consider the following integral:

  $$\int_\gamma \frac{dw}{(w-a)(w-z)} = \frac{1}{z-a}\int_\gamma \left(\frac{1}{w-a} - \frac{1}{w-z}\right) dw = 0.$$

  The last equality follows from Cauchy's formula for the values of the function 1 at a and z. Hence the last term in the above sum is 0. However the previous item implies that all the rest are 0 inside the circle. Conclude that:

  $$f_{N+1}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_(w)dw}{(w-z)(w-a)^{N+1}}.$$

- We will find a bound on $|f_{N+1}(z)(z-a)^{N+1}|$ independent of z.

  $$|f_{N+1}(z)(z-a)^{N+1}| \leq \frac{1}{2\pi} r^{N+1} \left| \int_\gamma \frac{f_(w)dw}{(w-z)(w-a)^{N+1}} \right| \leq r^{N+1}\frac{RM}{R^{N+1}(R-r)}.$$

  Here M is the bound of f on γ. This clearly goes to 0 an N goes to infinity.

- The previous item implies that the series converges uniformly in every closed subdisc of $D_R(a)$. Now R can be chosen arbitrarily close to the shortest distance from a to the boundary of Ω, this means that for every disc properly contained in Ω, the series will converge. Thus it will converge on the largest disc around a contained in Ω (albeit non-uniformly).

8. The idea in all those integrals it to substitute the following expressions for trigonometric functions and transform the integral into an integral on a circle:

   $$\cos(t) = \frac{e^{it} + e^{-it}}{2},\ \sin(t) = \frac{e^{it} + e^{-it}}{2}.$$

   Then we apply Cauchy's formula. Remember, we are integrating a real-valued function on the reals, so the result must be real!

- Compute:

$$\int_0^{2\pi} \frac{dx}{5+\cos(x)} = \int_0^{2\pi} \frac{dx}{5+\frac{e^{ix}+e^{-ix}}{2}} = 2\int_0^{2\pi} \frac{dx}{10+e^{ix}+\frac{1}{e^{ix}}} =$$
$$= 2\int_0^{2\pi} \frac{e^{ix}dx}{10e^{ix}+e^{2ix}+1} = \frac{2}{i}\int_0^{2\pi} \frac{ie^{ix}dx}{10e^{ix}+e^{2ix}+1} = \frac{2}{i}\int_{|z|=1} \frac{dz}{10z+z^2+1} =$$
$$= \frac{2}{i}\int_{|z|=1} \frac{dz}{(z-(-5+2\sqrt{6}))(z-(-5-2\sqrt{6}))} = \frac{2}{i}2\pi i\frac{1}{4\sqrt{6}} = \frac{\pi}{\sqrt{6}}.$$

- Here we use $\cos^2(x) = \frac{1+cos(2x)}{2}$ and $\sin^2(x) = \frac{1-cos(2x)}{2}$:

$$\int_0^{\pi} \frac{\cos^2(x)dx}{1-a\sin^2(x)} = \int_0^{\pi} \frac{(1+\cos(2x))dx}{2-a(1-\cos(2x))} = [y=2x] = \frac{1}{2}\int_0^{2\pi} \frac{(1+\cos(y))dy}{2-a+a\cos(y)} =$$
$$= \frac{1}{2a}\int_0^{2\pi} \frac{2-a-2+2a+a\cos(y)}{2-a+a\cos(y)}dy = \frac{1}{2a}\int_0^{2\pi}\left(1+\frac{2a-2}{2-a+a\cos(y)}\right)dy =$$
$$= \frac{\pi}{a}+\frac{a-1}{a}\int_0^{2\pi}\frac{dy}{2-a+a\cos(y)} = \frac{\pi}{a}+2\frac{a-1}{a}\int_0^{2\pi}\frac{dy}{4-2a+ae^{iy}+ae^{-iy}} =$$
$$= \frac{\pi}{a}+2\frac{a-1}{ia}\int_0^{2\pi}\frac{ie^{iy}dy}{(4-2a)e^{iy}+ae^{2iy}+a} = \frac{\pi}{a}+2\frac{a-1}{ia}\int_{|z|=1}\frac{dz}{(4-2a)z+az^2+a}.$$

  The roots of the denominator are $z_\pm = \frac{a-2\pm2\sqrt{1-a}}{a}$. Note that by the assumption on a, this number is real. The question is then which roots lie inside the unit disc? If $-1 < \frac{a-2\pm2\sqrt{1-a}}{a} < 1$, then $-a < a-2\pm2\sqrt{1-a} < a$, which in turn implies that $1-a < \pm\sqrt{1-a} < 1$. This is obviously false for the negative sign and true for the positive. Thus we can write the denominator as: $(4-2a)z+az^2+a = a(z-z_+)(z-z_-)$ We write $f(z) = \frac{1}{z-z_-}$. Now Cauchy formula tells us that:

$$\int_0^{\pi}\frac{\cos^2(x)dx}{1-a\sin^2(x)} = \frac{\pi}{a}+\pi\frac{a-1}{a}\frac{1}{\sqrt{1-a}} = \pi\frac{1-\sqrt{1-a}}{a}$$

- First note that since the integrand is even we get that:

$$\int_0^{\pi}\frac{dx}{(a+b\cos(x))^2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dx}{(a+b\cos(x))^2}$$

  Now compute:

$$\frac{1}{2}\int_{-\pi}^{\pi}\frac{dx}{(a+b\cos(x))^2} = 2\int_{-\pi}^{\pi}\frac{dx}{(2a+be^{ix}+be^{-ix})^2} = 2\int_{-\pi}^{\pi}\frac{e^{2ix}dx}{(2ae^{ix}+be^{2ix}+b)^2} =$$
$$= \frac{2}{i}\int_{|z|=1}\frac{zdz}{(2az+bz^2+b)^2}.$$

  The roots of the denominator are $z_\pm = \frac{-a\pm\sqrt{a^2-b^2}}{b}$. Note that those numbers are real by the assumption on a and b. Those numbers are inside the unit disc if $-b < -a\pm\sqrt{a^2-b^2} < b$, which implies $a-b < \pm\sqrt{a^2-b^2} < a+b$. Since $a-b>0$, z_- will not do, now squarring both sides we see that z_+ always satisfies this inequality. So write $f(z) = \frac{z}{(z-z_-)^2}$. This function is holomorphic in the unit disc and we get that:

$$\int_0^{\pi}\frac{dx}{(a+b\cos(x))^2} = \frac{2}{i}\int_{|z|=1}\frac{f(z)dz}{(z-z_+)^2} = 4\pi f'(z_+).$$

Now $f'(z) = \frac{(z-z_-)^2 - 2z(z-z_-)}{(z-z_-)^4} = \frac{(z-z_-)-2z}{(z-z_-)^3} = -\frac{z+z_-}{(z-z_-)^3}$. Plugging in z_+, we get:

$$f'(z_+) = -\frac{z_+ + z_-}{(z_+ - z_-)^3} = \frac{2\frac{a}{b}}{(\frac{2\sqrt{a^2-b^2}}{b})^3} = \frac{ab^2}{4(\sqrt{a^2-b^2})^3}.$$

9. Given such f and g write $1 = f^2(z) + g^2(z) = (f(z) + ig(z))(f(z) - if(z))$. This implies that $f + ig$ is non-vanishing on $\mathbb{C}$. Since f and g are entire, so is $h = f + ig$. Therefore $\frac{h'}{h}$ is also entire and since $\mathbb{C}$ itself is simply connected, it has a primitive function, H. Note that $e^H = h$. Write $H = iF$. Now we have that:

$$f - ig = (f + ig)^{-1} = e^{-iF}.$$

Now $f = \frac{(f+ig)+(f-ig)}{2} = \frac{e^{iF} - e^{-iF}}{2} = \cos(F)$. Similarly $g = \sin(F)$.

10.
   - The function f is holomorphic in Ω and thus have a primitive, since Ω is simply connected. Take the branch of the logarithm obtained on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and denote it by log. Then it is easy verify directly, that the primitive is given by $\frac{1}{2i}\log(\frac{z-i}{z+i})$. The only thing to note here is that the function $\phi(z) = \frac{z-i}{z+i}$ takes the lines it and $-it$, for $t \geq 1$ real, to the line $bbR_{\geq 0}$. This is the reason for the choice of the branch of the logarithm.
   - Note that on the real axis we have that $F(x) = \arctan(x)$, in particular on $(-\pi/2, \pi/2)$, we have that $F(\tan(x)) = x$. Define $h = F(\tan(z)) - z$, then h is holomorphic precisely in U as a composition of two holomorphic function (tan is holomorphic everywhere outside $\pi/2 + \pi k$). In particular $(-\pi/2, \pi/2) \subset U$ and h vanishes on this segment. By the uniqueness theorem it implies that f is identically 0 on U.

11. Note that $zf(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} z^{3n+1}$, take the derivative and get that $(zf(z))' = \sum_{n=0}^{\infty}(-1)^n z^{3n} = \sum_{n=0}^{\infty}(-1)^{3n} z^{3n} = \frac{1}{1+z^3}$. Now multiplying a power series by z does not change the radius of convergence, neither does taking derivative, the last series however converges in the unit disc. One can also check it directly with Cauchy-Hadamard's theorem.

   Let $g(z) = \frac{1}{1+z^3}$, it is holomorphic everywhere except for $z_k = e^{\frac{\pi i + 2\pi i k}{3}}$, $k = 0, 1, 2$. Now clearly if we find an open set containing the unit disc where g has a primitive, G, then on the unit disc $G(z) - zf(z)$ is constant (since the derivative is identicaly 0. So take G, such that $G(0) = 0$ and then $G(z) = zf(z)$ on the unit disc. Hence $F(z) = \frac{G(z)}{z}$ (which is holomorphic, since $G(z) = 0$). So it remains to find U. Clearly $\mathbb{C} \setminus \{z_1, z_2, z_3\}$ won't do, since the integral of g on a circle around just z_1 is, by Cauchy's formula, $\frac{2\pi i}{(z_1 - z_2)(z_1 - z_3)} \neq 0$. Let $L_k = \{tz_k \mid t \geq 1\}$ a line starting at z_k with angle $\frac{\pi + 2\pi k}{3}$. Let $U = \mathbb{C} \setminus (L_1 \bigcup L_2 \bigcup L_3)$. Then U is simply connected and g is holomorphic in U and thus has a primitive.